

# How the Pauli principle governs the decay of three-cluster systems

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## Abstract

New approach to the problem of multichannel continuum spectrum of three-cluster systems composed of an  $s$ -cluster and two neutrons is suggested based on the discrete representation of a complete basis of allowed states of the multiparticle harmonic oscillator. The structure of the eigenfunctions and behavior of the eigenvalues of the three-cluster norm kernel are analyzed. Classification of the eigenvalues of the three-cluster systems with the help of eigenvalues of the two-body subsystem is suggested. Asymptotic boundary conditions for a three-cluster wave function in the continuum consistent with the requirements of the Pauli principle are established. Such asymptotic behavior corresponds rather to subsequent decay of the three-cluster system than to the so-called "democratic decay" associated with the hyperspherical harmonics. The  ${}^3\text{H} + n + n$  configuration of the  ${}^5\text{H}$  nucleus is considered in detail.

*Key words:* three-cluster microscopic model, Pauli-allowed states, resonating-group method, neutron-rich nuclei  
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## 1. Introduction

During last years the superheavy hydrogen isotope  ${}^5\text{H}$  has become an object of numerous experimental [1,2,3,4,5] and theoretical [6,7,8,9,10] researches. The overwhelming majority of these investigations aimed at finding the energy and width of the  ${}^5\text{H}$  resonance states. Unfortunately, even for the  ${}^5\text{H}$  ground-state the experimental values for energy and width are significantly distinct and vary from  $E_R = 1.8 \pm 0.1$  MeV with  $\Gamma \leq 0.5$  MeV [2] to  $E_R = 5.5 \pm 0.2$  MeV with  $\Gamma = 5.4 \pm 0.5$  MeV [5]. Detailed discussion of the available experimental data is given in Ref. [11] with the conclusion that it is very difficult to find a non-contradictory scenario for all experimental data. Theoretical predictions are also different: in Refs. [8,9,10] a resonance  $J^\pi = 1/2^+$  is observed at  $E_R \simeq 1.5$  MeV, while in Refs. [6,7] nearly twice that energy was obtained. As for the model, the most commonly used approaches treat the  ${}^5\text{H}$  nucleus as a three-cluster system composed of a triton and two neutrons. Such assumption is justified by the fact that  ${}^3\text{H}$  is bound by 8.48 MeV [12], whereas  ${}^2\text{n}$  and  ${}^4\text{H}$  subsystems are known to be unbound. Mainly, different three-cluster models fall into two groups – macroscopic and microscopic. In macroscopic models clusters are considered to be structureless particles and cluster-cluster interactions are approximated by some local potentials which are fitted to reproduce relevant data on the cluster-cluster systems. As for the Pauli exclusion principle, it is usually simulated either with an additional repulsive potential between clusters or with orthogonalizing pseudopotential containing the operators of projection onto the forbidden states. The first approach was

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used in Refs. [6,9], where the blocking of the Pauli-forbidden states was realized by introducing a repulsive core in the s-wave  $^3\text{H}$ -neutron potentials. However, the choice of such a repulsive potential is quite ambiguous and within this approximation a complete and accurate exclusion of the forbidden states is not ensured. Furthermore, macroscopic three-cluster model rests on the assumption that cluster-cluster interactions are not affected by the presence of the third cluster, but this is by no means always the case. As regards the elimination of the Pauli-forbidden states with the help of orthogonalizing pseudopotential [13], it could result in exclusion of some Pauli-allowed states. For example, in Ref. [14] it was shown that some of the states of  $3\alpha$ -system obtained by the above-mentioned method could be regarded as spurious, which they were not. Moreover, the latter states correspond to the most important shell-model configurations of the  $^{12}\text{C}$  nucleus. Hence, the Pauli-allowed space should be carefully defined, in order not to exclude the dominant components for the realistic description of the three-cluster systems. For this purpose, one can define the three-cluster forbidden states referring to the harmonic oscillator wave functions of the microscopic two-cluster subsystem, as was suggested in Ref. [14]. This idea rested on the hypothesis that the antisymmetric three-cluster wave function should be orthogonal to the forbidden states of any two-cluster subsystem [15]. This method seems to cope well with the task of accurate removal of the Pauli-forbidden states in a three-cluster system. However, it does not allow to distinguish the Pauli-allowed functions belonging to the same  $\text{SU}(3)$ -multiplet, i.e. the well-known problem of  $\text{SU}(3)$ -degeneracy of three-cluster states could not be resolved within such approach. Moreover, the elimination of the Pauli-forbidden states does not exhaust all exchange effects. Essential part of such effects is directly relevant to the eigenvalues of the antisymmetrization operator. The latter eigenvalues are not identical and determine the realization probability of the corresponding Pauli-allowed basis states in the wave function of the cluster system. Involvement of the eigenvalues of the antisymmetrization operator in the Schrödinger equation leads to changing the relative kinetic energy as clusters approach each other. Consequently, clusters are shown to experience an effective repulsion or attraction arising from the kinetic energy operator modified by the Pauli principle [16]. Such an effective interaction substantially affects the dynamics of the cluster-cluster interaction and can, on occasion, produce resonance behavior of the scattering phase or even a bound state in compound nuclear system [17]. As was shown in Refs. [16,17], the largest eigenvalues correspond to those basis states that are dominant in the discrete-spectrum states of the binary cluster system and in continuum states of small above-threshold energy. Also in Ref. [18] it was observed that the probability of the presence of a cluster configuration in the lowest basis function for a binary cluster system is proportional to the eigenvalue of the isolated configuration. Hence, the eigenvalues of the Pauli-allowed states contain a wealth of information on a compound system composed of clusters. Meanwhile, the eigenvalues, along with corresponding Pauli-allowed states, depend only on the assumed internal cluster functions, not on the cluster-cluster potential, etc. Up to our knowledge, analysis of the structure of the eigenfunctions of the antisymmetrizer and behavior of its eigenvalues has never been performed for three-cluster systems, although we believe that it could help to establish some important laws that govern a three-cluster decay of nuclei composed of  $s$ -clusters.

The main difficulty in studies of resonances in a three-cluster system consists in the formulation of the correct asymptotic boundary conditions for a wave function in the continuum. Such boundary conditions should ensure continuous transition from the region of small intercluster distances, where the Pauli exclusion principle is of first importance, to the asymptotic region where the scattering matrix elements are produced. Among currently available microscopic studies of  $^5\text{H}$  [7,8,10], an explicit wave function representation of the scattering states have been employed only in Ref. [10]. In Ref. [7] the method of analytic continuation in the coupling constant was used for study of resonance states in  $^5\text{H}$ , while in Ref. [8] the complex scaling method was applied for the same purpose. It should be noted that although authors of [10] performed calculations within the same model as we do, namely within an Algebraic Version of the resonating-group method (AVRGM) [19], our approaches essentially differ in such aspects. To classify the three-cluster states, and enumerate the channels in the three-cluster continuum, hyperspherical harmonics were used in Ref. [10], while we define the Pauli-allowed harmonic oscillator basis functions in the Fock-Bargmann space and classify them with the use of the  $\text{SU}(3)$  symmetry indices. The latter seems more appropriate, because the second-order Casimir operator of the  $\text{SU}(3)$  group commutes with the operator of permutation of the nucleon position vectors. Hence, the  $\text{SU}(3)$  symmetry indices naturally appear as the quantum numbers of the eigenfunctions of the antisymmetrization operator. The hypermomentum, contrastingly, can not serve

as quantum number of Pauli-allowed basis functions, because the Pauli exclusion principle mixes the states with different values of hypermomentum. Hence, restriction for the maximum value of hypermomentum  $K$  included in calculation  $K \leq K_{max}$  leads to spoiling the Pauli-allowed basis functions corresponding to the number of oscillator quanta  $\nu > K_{max}$ , and this effect is enhanced with increasing  $\nu$ . In support of this conjecture it was observed in Ref. [20] that an accurate description of the three-body asymptotics requires bases with large hypermomenta.

In the present paper we have shown that correct asymptotic boundary conditions can be found employing a complete basis of the Pauli-allowed harmonic-oscillator states (classified with the use of the SU(3) symmetry indices and defined in the Fock-Bargmann space) along with their eigenvalues. Asymptotic behavior of basis functions consistent with the requirements of the Pauli principle gives an indication of possible decay channels of a three-cluster nucleus and allows us to specify the most important decay channels. Such asymptotic behavior correspond rather to subsequent decay of the three-cluster system than to the so-called "democratic decay" associated with the hyperspherical harmonics, which are widely used for the description of three-cluster systems (see Refs. [6,9,10]).

In Section 2 the theoretical grounds of the AVRGM are outlined and the structure of the norm kernel for the number of three-cluster systems composed of  $s$ -cluster and two neutrons is discussed. Section 3 is devoted to the analysis of the eigenvalues and eigenfunctions of the norm kernel of the  ${}^A\text{X} = {}^{A-2}\text{X} + n + n$  systems ( $A \leq 6$ ). The asymptotic behavior of the coefficients of the expansion of the three-cluster continuum wave function in the SU(3) basis is established in Section 4. In Section 5 the Pauli-allowed states of the  ${}^5\text{H}={}^3\text{H}+n+n$  system are considered in detail. The most important decay channels of the  ${}^5\text{H}$  nucleus are specified and illustrated with the results of numerical calculations. Concluding remarks are given in Section 6. Definition and some properties of the Kravchuk polynomials are given in Appendix A.

## 2. Theoretical background and norm kernel

Following the resonating-group method (RGM) [21], it will hereafter be supposed that the considered nuclear systems consist of three clusters. An RGM wave function is built in the form of an antisymmetrized product of cluster internal wave functions and a wave function of their relative motion. The internal wave functions of the clusters are fixed<sup>1</sup>, and the wave function of relative motion of the clusters, which depends only on two Jacobi vectors of the considered three-cluster system, is found by solving an integro-differential equation. The latter is obtained by substitution of the RGM wave function into the Schrödinger equation followed by integration with respect to single-particle coordinates. The integro-differential equation can be transformed into the set of linear equations by expanding the wave function of the cluster relative motion into the compete basis of the Pauli-allowed harmonic oscillator states, as the AVRGM suggests. Another important simplification can be achieved by transformation from the coordinate space to the space of complex generator parameters (the Fock–Bargmann space [22]), in which basis functions are of an especially simple form and are expressed via powers of complex vectors. Thus, the wave functions of the considered discrete representation are reduced to power series with an infinitely large convergence radius. The validity of this statement is indicated by the fact that all the wave functions in the Fock-Bargmann space are entire and analytic, and, therefore, series of these functions over powers of complex vectors converge in any finite region of the complex space.

First and foremost, the AVRGM calls for the construction of the complete basis of the Pauli-allowed harmonic oscillator states and their classification. This is accomplished by solving the eigenvalue and eigenfunction problem for the norm kernel, i.e., the overlap integral of the two Slater determinants composed of the single-particle orbitals:

$$I(\{\mathbf{S}_j\}, \{\mathbf{R}_j\}) = \int \Phi(\{\mathbf{S}_j\}, \mathbf{r}) \Phi(\{\mathbf{R}_j\}, \mathbf{r}) d\tau. \quad (1)$$

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<sup>1</sup> Here we shall assume the intrinsic cluster wave functions to be the simplest functions of a translation-invariant shell model

Here integration is performed over all single-particles vectors,  $\{\mathbf{R}_j\}$  (or  $\{\mathbf{S}_j\}$ ) identifies the set of three complex vectors determining position of the center-of-mass of clusters in the Fock-Bargmann space. For the spatial part of the single-particle wave functions we used the modified Bloch-Brink orbitals:

$$\phi(\mathbf{r}_i) = \frac{1}{\pi^{3/4}} \exp\left(-\frac{1}{2}\mathbf{r}_i^2 + \sqrt{2}(\mathbf{R}_j \cdot \mathbf{r}_i) - \frac{1}{2}\mathbf{R}_j^2\right), \quad i \in A_j,$$

where  $A_j$  is the number of nucleons in the  $j$ th cluster. Each of these orbitals is an eigenfunction of the coordinate operator  $\hat{\mathbf{r}}_i$ :

$$\hat{\mathbf{r}}_i = \frac{1}{\sqrt{2}} \left( \mathbf{R}_j + \vec{\nabla}_{\mathbf{R}_j} \right); \quad \mathbf{R}_j = \frac{\vec{\xi}_j + i\vec{\eta}_j}{\sqrt{2}}.$$

which is defined in the Fock-Bargmann space and corresponds to the eigenvalue  $\mathbf{r}_i$ .  $\vec{\xi}_j$  and  $\vec{\eta}_j$  are vectors of coordinate and momentum, respectively. At the same time, orbital  $\phi(\mathbf{r}_i)$  is the kernel of the integral transform from the coordinate representation to the Fock-Bargmann representation [22] and the generating function for the harmonic-oscillator basis [23]. The center-of-mass motion are factored out (and dropped out from now on) by transition from generator parameters  $\{\mathbf{R}_j\}$  to the Jacobi vectors:

$$\mathbf{R}_{cm} = \frac{1}{\sqrt{A}} (A_1 \mathbf{R}_1 + A_2 \mathbf{R}_2 + A_3 \mathbf{R}_3),$$

$$\mathbf{a} = \sqrt{\frac{A_1(A_2 + A_3)}{A}} \left( \mathbf{R}_1 - \frac{A_2 \mathbf{R}_2 + A_3 \mathbf{R}_3}{A_2 + A_3} \right), \quad \mathbf{b} = \sqrt{\frac{A_2 A_3}{A_2 + A_3}} (\mathbf{R}_2 - \mathbf{R}_3).$$

As the result, the overlap of two Slater determinants composed of the modified Bloch-Brink orbitals generates a complete basis of the Pauli-allowed harmonic-oscillator functions along with their eigenvalues:

$$I(\{\mathbf{S}_j\}, \{\mathbf{R}_j\}) \Rightarrow I(\mathbf{a}, \mathbf{b}; \tilde{\mathbf{a}}, \tilde{\mathbf{b}}) = \sum_n \Lambda_n \Psi_n(\mathbf{a}, \mathbf{b}) \Psi_n(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}).$$

Functions  $\Psi_n$  are defined in the Fock-Bargmann space and orthonormalized with the Bargmann measure  $d\mu_B$ :

$$d\mu_B = \exp\{-(\mathbf{a}\tilde{\mathbf{a}})\} \frac{d\vec{\xi}_a d\vec{\eta}_a}{(2\pi)^3} \exp\{-(\mathbf{b}\tilde{\mathbf{b}})\} \frac{d\vec{\xi}_b d\vec{\eta}_b}{(2\pi)^3},$$

$n$  stands for the set of quantum numbers of basis functions. Quantum numbers and the structure of functions  $\Psi_n$  will be discussed in the next section.

Examine next the explicit expressions for the norm kernels of those three-cluster systems which comprised of an  $s$ -cluster and two neutrons, such as  ${}^6\text{He}={}^4\text{He}+n+n$ ,  ${}^5\text{H}={}^3\text{H}+n+n$ ,  ${}^4n={}^2n+n+n$ ,  ${}^3n=n+n+n$ , subsequently referred to in this article as  ${}^A\text{X}={}^{A-2}\text{X}+n+n$ ,  $A \leq 6$ . The norm kernels for all these systems can be written in general form:

$$I_{S_{(nn)}=0} = \exp(\mathbf{a}\tilde{\mathbf{a}}) \cosh(\mathbf{b}\tilde{\mathbf{b}}) - \exp(\mathbf{a}_+\tilde{\mathbf{a}}) \cosh(\mathbf{b}_+\tilde{\mathbf{b}}) - \exp(\mathbf{a}_-\tilde{\mathbf{a}}) \cosh(\mathbf{b}_-\tilde{\mathbf{b}}) + \exp\{(\mathbf{a}_0\tilde{\mathbf{a}})\}, \quad (2)$$

$$I_{S_{(nn)}=1} = \exp(\mathbf{a}\tilde{\mathbf{a}}) \sinh(\mathbf{b}\tilde{\mathbf{b}}) - \exp(\mathbf{a}_+\tilde{\mathbf{a}}) \sinh(\mathbf{b}_+\tilde{\mathbf{b}}) - \exp(\mathbf{a}_-\tilde{\mathbf{a}}) \sinh(\mathbf{b}_-\tilde{\mathbf{b}}) + \exp\{(\mathbf{a}_0\tilde{\mathbf{a}})\}. \quad (3)$$

Here  $S_{(nn)}$  designates the spin of the two-neutron subsystem. At that the so-called "T-tree" of the Jacobi coordinates is considered, i.e., vector  $\mathbf{b}$  describes relative distance between two neutrons, while vector  $\mathbf{a}$  indicates the distance from the remaining cluster to the center of mass of the two-neutron subsystem.

Note that  $I_{S_{(nn)}=0}$  is symmetric with respect to the permutation of two valence neutrons, while  $I_{S_{(nn)}=1}$  is antisymmetric as it must.

Each term in Eqs. (2) and (3) corresponds to a certain permutation of identical nucleons. The first exponent is associated with the identity permutation, the second and the third exponents appeared as the result of permutation of a valence neutron and a neutron belonging to the  $s$ -cluster, while the last term relates to the simultaneous permutation of two pairs of identical neutrons:

$$\mathbf{a}_\pm = \left(1 - \frac{t}{2}\right) \mathbf{a} \pm \frac{1}{2} \sqrt{t} \mathbf{b}, \quad \mathbf{b}_\pm = \pm \frac{1}{2} \sqrt{t} \mathbf{a} + \frac{1}{2} \mathbf{b}, \quad \mathbf{a}_0 = (1-t) \mathbf{a}. \quad (4)$$

Finally, the parameter  $t$  is positive and equal to the square of tangent of the angle of rotation from one Jacobi tree to another.  $t$  takes the value  $3/2$  for  ${}^6\text{He}$  nucleus,  $5/3$  for the  ${}^5\text{H}$ ,  $t = 2$  for tetraneutron and  $t = 3$  for three-neutron system. As can readily be observed, parameter  $t$  decreases with increasing the number of nucleons in  $s$ -cluster.

The eigenvalues of the norm kernel do not depend on the choice of the Jacobi tree, whereas the structure of the eigenfunctions does. Since the Pauli-allowed basis states of the  ${}^A\text{X} = {}^{A-2}\text{X} + n + n$  systems take the simplest form in the "T-tree", it precisely this tree is best suited to the construction and analysis of these states.

### 3. Eigenvalues and eigenfunctions of the norm kernel of the ${}^A\text{X} = {}^{A-2}\text{X} + n + n$ systems

Let us discuss now the set of quantum numbers of the Pauli-allowed states  $\Psi_n(\mathbf{a}, \mathbf{b})$ . For the three-cluster systems considered here,  $n$  includes the number of oscillator quanta  $\nu$ , the indices  $(\lambda, \mu)$  of their SU(3) symmetry, the additional quantum number  $k$  if there are several differing  $(\lambda, \mu)$  multiplets, the orbital momentum  $L$  and its projection  $M$  and, if necessary, one more additional quantum number  $\alpha_L$ . The latter is needed to label the states with the same  $L$  in a given  $(\lambda, \mu)$  multiplet. The spin of the two-neutron subsystem  $S_{(nn)}$  also is an integral of motion for as long as the spin-orbital interaction is switched off. It is well known that the diagonalization of the norm kernel requires the basis to be labeled with the quantum indices  $(\lambda, \mu)$  of irreducible representations of the SU(3) group [24]. The eigenvalues of the norm kernel depend on the total number of the oscillator quanta and  $(\lambda, \mu)$ , and do not depend on the angular momenta of the basis states. We shall restrict our consideration to the states with zero orbital momentum  $L = 0$  and positive parity. Hence, the number of oscillator quanta should be even and equal  $2\nu$ , quantum numbers  $L$  and  $M$  will be dropped from now on.

By definition, the states  $\Psi_n(\mathbf{a}, \mathbf{b})$  are the eigenfunctions of the antisymmetrization operator  $\hat{A}$ :

$$\hat{A} \Psi_{(2\nu-4\mu, 2\mu)_k}^{S_{(nn)}} = \Lambda_{(2\nu-4\mu, 2\mu)_k}^{S_{(nn)}} \Psi_{(2\nu-4\mu, 2\mu)_k}^{S_{(nn)}}.$$

In two-cluster systems the eigenvalues of antisymmetrizer limit to unity at  $\nu \rightarrow \infty$  and the deviations from the unity are due to the Pauli exclusion principle. Contrastingly, eigenvalues of three-cluster systems tend to eigenvalues of a two-cluster subsystem with increasing the number of oscillator quanta  $\nu$ :

$$\lim_{\nu-2\mu \rightarrow \infty} \Lambda_{(2\nu-4\mu, 2\mu)_k}^{{}^A\text{X} = {}^{A-2}\text{X} + n + n} \rightarrow \lambda_k^{{}^{A-1}\text{X} = {}^{A-2}\text{X} + n}.$$

Coincidence of the eigenvalues of the three-cluster norm kernels with those of the two-cluster subsystem norm kernels in the limit of the large number of quanta has been already pointed out in [26], where  $3\alpha$  and  ${}^{16}\text{O}+2\alpha$  systems have been considered. However, no relation with the additional quantum number  $k$  labeling the states with the same SU(3)-indices has been established. This subject will be pursued further in the next section. Now let us proceed with the Pauli-allowed basis functions  $\Psi_{(2\nu-4\mu, 2\mu)_k}^{S_{(nn)}}(\mathbf{a}, \mathbf{b})$ .

In the Fock-Bargmann space the latter functions are the superpositions of the eigenfunctions of the second-order Casimir operator:

$$\Psi_{(2\nu-4\mu, 2\mu)_k}^{S_{(nn)}=0}(\mathbf{a}, \mathbf{b}) = \sum_{m=\mu}^{\nu-\mu} D_{(2\nu-4\mu, 2\mu)_k}^{2m-2\mu} \psi_{(2\nu-4\mu, 2\mu)}^{2m-2\mu}(\mathbf{a}, \mathbf{b}),$$

$$\Psi_{(2\nu-4\mu,2\mu)_k}^{S_{(nn)}=1}(\mathbf{a}, \mathbf{b}) = \sum_{m=\mu}^{\nu-\mu-1} D_{(2\nu-4\mu,2\mu)_k}^{2m-2\mu+1} \psi_{(2\nu-4\mu,2\mu)}^{2m-2\mu+1}(\mathbf{a}, \mathbf{b}),$$

Functions  $\psi_{(2\nu-4\mu,2\mu)}^{2m-2\mu}(\mathbf{a}, \mathbf{b})$  and  $\psi_{(2\nu-4\mu,2\mu)}^{2m-2\mu+1}(\mathbf{a}, \mathbf{b})$  correspond to the same total number of oscillator quanta  $2\nu$ , but differ in the number of quanta along vectors  $\mathbf{a}$  and  $\mathbf{b}$ . That is the reason why the three-cluster states are always degenerate, even though only  $s$ -clusters involved, and the degree of the SU(3) degeneracy increases drastically with increasing the number of quanta. It was found in Ref. [25] that the eigenfunctions of the Casimir operator with powers of vectors  $\mathbf{a}$  and  $\mathbf{b}$  being fixed are expressible in terms of hypergeometric functions  ${}_2F_1(\alpha, \beta; \gamma; Z)$ , with the variable

$$Z = \frac{[\mathbf{ab}]^2}{\mathbf{a}^2 \mathbf{b}^2}.$$

Namely,

$$\begin{aligned} \psi_{(2\nu-4\mu,2\mu)}^{2m-2\mu}(\mathbf{a}, \mathbf{b}) &= N_{(2\nu-4\mu,2\mu)}^{2m-2\mu} [\mathbf{ab}]^{2\mu} \mathbf{a}^{2\nu-2m-2\mu} \mathbf{b}^{2m-2\mu} \times \\ &\quad \times {}_2F_1 \left( -\nu + m + \mu, -m + \mu; -\nu + 2\mu + \frac{1}{2}; Z \right), \\ \psi_{(2\nu-4\mu,2\mu)}^{2m-2\mu+1}(\mathbf{a}, \mathbf{b}) &= N_{(2\nu-4\mu,2\mu)}^{2m-2\mu+1} [\mathbf{ab}]^{2\mu} (\mathbf{ab})^{2\nu-2m-2\mu-2} \mathbf{a}^{2m-2\mu} \times \\ &\quad \times {}_2F_1 \left( -\nu + m + \mu + 1, -m + \mu; -\nu + 2\mu + \frac{1}{2}; Z \right). \end{aligned}$$

Here  $N_{(2\nu-4\mu,2\mu)}^{2m-2\mu}$  and  $N_{(2\nu-4\mu,2\mu)}^{2m-2\mu+1}$  are the normalization coefficients of the corresponding basis functions.

The norm kernel  $I_{S_{(nn)}}$  can be written down as a sum of partial norm kernels with definite values of the number of quanta  $2\nu$ :

$$I_{S_{(nn)}=0} = \sum_{\nu=0}^{\infty} I_{S_{(nn)}=0}^{2\nu}, \quad I_{S_{(nn)}=1} = \sum_{\nu=1}^{\infty} I_{S_{(nn)}=1}^{2\nu}.$$

In the Fock–Bargmann representation, at a given  $\nu$  the norm kernel  $I_{S_{(nn)}}^{2\nu}$  is always representable in the form of a sum of  $SU(3)$ -projected norm kernels  $I_{S_{(nn)}}^{(\lambda,\mu)}$ . We shall deal with the relevant part of the norm kernel,  $I_{S_{(nn)}}^{(2\nu-4\mu,2\mu)}$  and write it as

$$\begin{aligned} I_{S_{(nn)}=0}^{(2\nu-4\mu,2\mu)} &= \sum_{m=\mu}^{\nu-\mu} \sum_{\tilde{m}=0}^{\nu-m} \langle 2\nu, 2m-2\mu | 2\nu, 2m-2\mu + 2\tilde{m} \rangle \times \\ &\quad \times \left( \psi_{(2\nu-4\mu,2\mu)}^{2m-2\mu} \tilde{\psi}_{(2\nu-4\mu,2\mu)}^{2m-2\mu+2\tilde{m}} + \psi_{(2\nu-4\mu,2\mu)}^{2m-2\mu+2\tilde{m}} \tilde{\psi}_{(2\nu-4\mu,2\mu)}^{2m-2\mu} \right), \\ I_{S_{(nn)}=1}^{(2\nu-4\mu,2\mu)} &= \sum_{m=\mu}^{\nu-\mu-1} \sum_{\tilde{m}=0}^{\nu-m-\mu-1} \langle 2\nu, 2m+1-2\mu | 2\nu, 2m+1-2\mu + 2\tilde{m} \rangle \times \\ &\quad \times \left( \psi_{(2\nu-4\mu,2\mu)}^{2m+1-2\mu} \tilde{\psi}_{(2\nu-4\mu,2\mu)}^{2m+1-2\mu+2\tilde{m}} + \psi_{(2\nu-4\mu,2\mu)}^{2m+1-2\mu+2\tilde{m}} \tilde{\psi}_{(2\nu-4\mu,2\mu)}^{2m+1-2\mu} \right), \end{aligned}$$

where

$$\begin{aligned} \langle 2\nu, 2m-2\mu | 2\nu, 2m-2\mu + 2\tilde{m} \rangle &= \delta_{\tilde{m},0} (1 + \delta_{m,0} \delta_{\mu,0} (1-t)^{2\nu}) - \\ &- \frac{2t^{\tilde{m}}}{4^{\nu-\mu}} \left( \frac{t-1}{t-2} \right)^{2\mu} (t-2)^{2(\nu-m-\tilde{m})} \sqrt{\frac{(2\nu-2m-2\mu)!(2m+2\tilde{m}-2\mu)!}{(2\nu-2m-2\tilde{m}-2\mu)!(2m-2\mu)!}} \times \\ &\quad \times {}_2F_1 \left( -2\nu + 2m + 2\mu + 2\tilde{m}, -2m + 2\mu; 2\tilde{m} + 1; \frac{t}{2-t} \right) \end{aligned}$$

$$\begin{aligned} \langle 2\nu, 2m+1-2\mu | 2\nu, 2m+1-2\mu + 2\tilde{m} \rangle &= \delta_{\tilde{m},0} + \\ + \frac{2t^{\tilde{m}}}{4^{\nu-\mu}} \left( \frac{t-1}{t-2} \right)^{2\mu} (t-2)^{2(\nu-m-\tilde{m})-1} \sqrt{\frac{(2\nu-2m-2\mu-1)!(2m+2\tilde{m}+1-2\mu)!}{(2\nu-2m-2\tilde{m}-1-2\mu)!(2m+1-2\mu)!}} \times \\ \times {}_2F_1 \left( -2\nu+2m+2\mu+1+2\tilde{m}, -2m-1+2\mu; 2\tilde{m}+1; \frac{t}{2-t} \right). \end{aligned}$$

Obviously, the Pauli-allowed basis functions  $\Psi_{(2\nu-4\mu,2\mu)_k}^{S_{(nn)}}$  can be found by the diagonalization of the norm kernels  $I_{S_{(nn)}}^{(2\nu-4\mu,2\mu)}$  at a given  $\nu$ . There are  $\nu-2\mu+1$  different states possessing SU(3) symmetry ( $2\nu-4\mu, 2\mu$ ), but some of them correspond to zero eigenvalues and thus they are forbidden by the Pauli principle. The number of the latter functions depends on the system considered. Hence, solving the eigenvalue and eigenfunction problem for the norm kernel  $I(\mathbf{a}, \mathbf{b}; \tilde{\mathbf{a}}, \tilde{\mathbf{b}})$  in the Fock-Bargmann space, we arrive at all the Pauli-allowed and the Pauli-forbidden functions of the  ${}^A X = {}^{A-2} X + n + n$  systems given in terms of orthogonal polynomials of a discrete variable.

As the eigenvalues  $\Lambda_{(2\nu-4\mu,2\mu)_k}^{A-2X=A-2X+n}$  of the norm kernel approach limit values  $\lambda_k^{A-1X=A-2X+n}$ , corresponding eigenvectors  $\Psi_{(2\nu-4\mu,2\mu)_k}^{S_{(nn)}}$  take simple analytical form:

$$\Psi_{(2\nu-4\mu,2\mu)_k}^{S_{(nn)}=0}(\mathbf{a}, \mathbf{b}) \rightarrow \frac{1}{\sqrt{2}} \left( \psi_{(2\nu-4\mu,2\mu)}^{k-2\mu}(\mathbf{a}_1, \mathbf{b}_1) + \psi_{(2\nu-4\mu,2\mu)}^{k-2\mu}(\mathbf{a}_2, \mathbf{b}_2) \right), \quad (5)$$

$$\Psi_{(2\nu-4\mu,2\mu)_k}^{S_{(nn)}=1}(\mathbf{a}, \mathbf{b}) \rightarrow \frac{1}{\sqrt{2}} \left( \psi_{(2\nu-4\mu,2\mu)}^{k-2\mu}(\mathbf{a}_1, \mathbf{b}_1) - \psi_{(2\nu-4\mu,2\mu)}^{k-2\mu}(\mathbf{a}_2, \mathbf{b}_2) \right). \quad (6)$$

Such asymptotical behavior takes place with the proviso that  $\nu \gg k$ , but as will be exemplified with  ${}^5 H$  system, the fulfilment of condition  $\nu \geq k+5$  would be ample. Note that  $k$  is the natural integer that falls in the range  $2\mu \leq k \leq \nu$ .

Here  $\mathbf{a}_1, \mathbf{b}_1$  and  $\mathbf{a}_2, \mathbf{b}_2$  are the Jacobi vectors of the so-called "Y-tree", with vector  $\mathbf{b}_1$  ( $\mathbf{b}_2$ ) describing relative distance between one of the valence neutron and  ${}^{A-2} X$ -cluster. Correspondingly, vector  $\mathbf{a}_1$  ( $\mathbf{a}_2$ ) determines position of the remaining neutron relative to the center-of-mass of the  ${}^{A-1} X$  subsystem. Jacobi vectors of the "T-tree" are related to those of the "Y-tree" via unitary transformation:

$$\mathbf{a} = \cos \alpha \mathbf{a}_{1,2} + \sin \alpha \mathbf{b}_{1,2}; \mathbf{b} = \mp \sin \alpha \mathbf{a}_{1,2} \pm \cos \alpha \mathbf{b}_{1,2}.$$

Naturally, function  $\Psi_{(2\nu-4\mu,2\mu)_k}^{S_{(nn)}=0}$  is symmetric with respect to the interchange of the vectors  $(\mathbf{a}_1, \mathbf{b}_1)$  and  $(\mathbf{a}_2, \mathbf{b}_2)$ , while  $\Psi_{(2\nu-4\mu,2\mu)_k}^{S_{(nn)}=1}$  is antisymmetric, in so far as such operation corresponds to the permutation of the valence neutrons.

The remarkable feature of asymptotical relations (5) and (6) lies in the fact that in the limit  $\nu \gg k$  the expansion coefficients  $D_{(2\nu-4\mu,2\mu)_k}^{2m-2\mu}, D_{(2\nu-4\mu,2\mu)_k}^{2m-2\mu+1}$  can be identified with the Kravchuk polynomials of a discrete variable [27]:

$$D_{(2n-4\mu,2\mu)_k}^{2m-2\mu} \rightarrow \sqrt{2} \mathcal{K}_{k-2\mu}^{(p)}(2m-2\mu) \frac{\sqrt{\rho_{2m-2\mu}}}{d_{k-2\mu}},$$

$$D_{(2n-4\mu,2\mu)_k}^{2m-2\mu+1} \rightarrow \sqrt{2} \mathcal{K}_{k-2\mu}^{(p)}(2m-2\mu+1) \frac{\sqrt{\rho_{2m-2\mu+1}}}{d_{k-2\mu}}, \quad p = \sin^2 \alpha.$$

The Kravchuk polynomials  $\mathcal{K}_{k-2\mu}^{(p)}(m-2\mu)$  are specified on the interval  $2\mu \leq m \leq 2\nu-2\mu$  and orthogonal with weighting function  $\rho_{m-2\mu}$  and norm  $d_{k-2\mu}$ . An explicit expression for the Kravchuk polynomials, as well as some of their basic properties, are given in Appendix A. Here it is worth noting that the Kravchuk polynomials are the discrete analog of the Hermitian polynomials.

So, the degree of the Kravchuk polynomials serves as the additional quantum number of the SU(3) degenerate three-cluster states. The Pauli-allowed basis states  $\Psi_{(2\nu-4\mu,2\mu)_k}^{S_{(nn)}}$  can be arranged into branches and families, with all the states of a particular branch having common symmetry index  $\mu$  and overlapping generously with corresponding asymptotic function (5) or (6), but differing in value of the first index  $\lambda$ . The eigenvalues belonging to a given branch tend to the same limit value  $\lambda_k^{A-1X}$  with the number of quanta increasing. The branches which share limit eigenvalues are combined in the family of the eigenstates, which thus is completely determined by the degree  $k$  of the corresponding Kravchuk polynomial.

Evidence for the importance of a particular family of the Pauli-allowed states can be found by analyzing the behavior of their eigenvalues with the increasing the number of oscillator quanta  $\nu$ . In the Introduction mention was made of the fact that the change in the kinetic energy of relative motion under the effect of the Pauli exclusion principle leads to an effective interaction of the nuclei as they approach each other. As is shown in Refs. [16,17], the character of such interaction in two-cluster systems is determined by the behavior of the eigenvalues of the antisymmetrization operator. They are nonnegative, since they are proportional to the probability of the realization of the corresponding allowed basis state, and tend to unity with increasing distance between the nuclei involved. In the case where the eigenvalues of the antisymmetrization operator tend to unity from below, the effective interaction proves to be repulsive, but, if these eigenvalues tend to unity from above, one can say that there arises attraction induced by the Pauli exclusion principle. As indicated earlier, the eigenvalues  $\Lambda_{(2\nu-4\mu,2\mu)_k}^{A_X=A^2X+n+n}$  of the three-cluster norm kernel approach the eigenvalues of the two-cluster subsystem  $\lambda_k^{A-1X}$  with increasing the number of quanta  $\nu$ . It would appear reasonable that families of the states with  $\Lambda_{(2\nu-4\mu,2\mu)_k} < \lambda_k$  would be suppressed at small  $\nu$ , which can naturally be considered as a demonstration of the action of effective forces of repulsion for small intercluster distances. Contrastingly, those families which characterized by  $\Lambda_{(2\nu-4\mu,2\mu)_k} > \lambda_k$  should become more favorable at small  $\nu$ , which can be considered as effective attraction. The validity of these assumptions for the  ${}^5H$  nucleus will be demonstrated in Section 5.

It is worth noting that the effective interaction caused by the change in the kinetic energy of the relative motion of clusters under the effect of the antisymmetrization operator arises only between objects whose internal energy may change when they approach each other<sup>2</sup>. Therefore, no such interaction would appear between two point-like nucleons. By this reason, the Pauli-allowed states in  ${}^3n=n+n+n$  system will most likely have the asymptotic behavior different from (5),(6).

#### 4. Asymptotic equations for the expansion coefficients

We seek the wave function of the considered three-cluster system in the form of expansion over the SU(3) basis of the Pauli-allowed states

$$\Upsilon_{\kappa(E)}(\mathbf{a}, \mathbf{b}) = \sum_n \sqrt{\Lambda_n} C_n^{\kappa(E)} \Psi_n(\mathbf{a}, \mathbf{b}). \quad (7)$$

Expansion coefficients both of the discrete eigenstates with energy  $E_\kappa = -\kappa^2/2 < 0$  and of the continuum eigenstates  $\{C_n(E)\}$  with energy  $E > 0$  are found by solving a set of linear equations

$$\sum_{n'} \langle n | \hat{H} | n' \rangle C_{n'} - E \Lambda_n C_n = 0. \quad (8)$$

In Ref. [28] we have shown that in order to set the asymptotic boundary conditions for the expansion coefficients of a two-cluster wave function in the SU(3) basis, a basis with a different set of quantum numbers (the angular-momentum coupled basis) is required. The states of the latter basis are labeled with angular momenta of the clusters and of their relative motion. The equations (8) for the expansion coefficients in the SU(3) basis remain coupled even in the asymptotic region, whereas the set of corresponding equations in the angular-momentum coupled basis is uncoupled at a large number of excitation quanta. The transformation

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<sup>2</sup> Here, we imply the interaction of nuclei consisting of nucleons.

between the two is defined through a unitary matrix and can be found with the use of the integration technique in the Fock–Bargmann space developed by the present authors. However, the norm kernel of binary cluster system has a diagonal form only in the representation of the basis of its eigenfunctions, i.e., in the representation of the SU(3)-basis. A unitary transform of the basis disrupts the diagonal form of the norm kernel due to the difference between the eigenvalues for different SU(3) representations. This would not be the case if all the eigenvalues were equal. The nature of this breaking is that, unlike the functions of the SU(3) basis, those of the angular-momentum coupled basis are not eigenfunctions of the antisymmetrization operator and, therefore, are not invariant with respect to a permutation of the nucleons. The permutation mixes the angular-momentum coupled basis functions with the same number of quanta. However, as the latter increases, the degree of mixing decreases exponentially, and at large values of the number of oscillator quanta the norm kernel becomes practically diagonal in the angular-momentum coupled basis as well. Hence, in this region the asymptotic behavior of the expansion coefficients in the angular-momentum coupled basis can be defined (they are expressed in terms of Hankel functions of the first and second kind, and the scattering  $S$ -matrix elements) and related to that of the expansion coefficients in the SU(3) basis.

Contrastingly to the eigenvalues of the two-cluster norm kernels, those of the three-cluster norm kernel remain distinct even in the limit  $\nu \rightarrow \infty$ . Existence of different limit eigenvalues  $\lambda_k$  reflects the possibility for two of three clusters to be close to each other and far apart from the third cluster. Owing to this the unitary transformation from the SU(3)-basis to the other one is rendered possible only within a particular family of the Pauli-allowed states. Any transformation, which involves the states belonging to different families (for example, transformation to the three-cluster hyperspherical harmonics), would disrupt the diagonal form of the norm kernel and, hence, is inappropriate to the occasion.

Turning back to the asymptotic SU(3) basis functions (5),(6), it might be well to point out that the latter functions have a simple physical meaning. They reproduce relative motion of a two-cluster subsystem  ${}^{A-1}X$  occurring in a ground or an excited harmonic-oscillator state and a remaining neutron. Such asymptotic behavior gives an indication of possible decay channels of a three-cluster nucleus and allows us to specify the most important decay channels of the nucleus under consideration.

In the limit  $\nu \gg k$  instead of asymptotical SU(3) basis functions

$$\frac{1}{\sqrt{2}} \left( \psi_{(2\nu-4\mu,2\mu)}^{k-2\mu}(\mathbf{a}_1, \mathbf{b}_1) \pm \psi_{(2\nu-4\mu,2\mu)}^{k-2\mu}(\mathbf{a}_2, \mathbf{b}_2) \right)$$

it is advantageous to use the angular momentum basis functions

$$\frac{1}{\sqrt{2}} (\phi_{2\nu,k}^l(\mathbf{a}_1, \mathbf{b}_1) \pm \phi_{2\nu,k}^l(\mathbf{a}_2, \mathbf{b}_2)),$$

defined as follows:

$$\phi_{2\nu,k}^l(\mathbf{a}, \mathbf{b}) = N_{\nu,k}^l \mathbf{a}^{2\nu-k-l} \mathbf{b}^{k-l} (\mathbf{ab})^l \cdot \frac{1}{2^l} \sum_{\tilde{l}=0}^{[l/2]} \frac{(-1)^{\tilde{l}} (2l - 2\tilde{l})!}{\tilde{l}!(l-\tilde{l})!(l-2\tilde{l})!} \frac{\mathbf{a}^{2\tilde{l}} \mathbf{b}^{2\tilde{l}}}{(\mathbf{ab})^{2\tilde{l}}}.$$

The states of this basis (referred to as "l-basis" in what follows) are labeled by the number of quanta  $2\nu$ , and the angular momenta  $l$  of the  ${}^{A-1}X$  subsystem coinciding with the angular momentum of the relative motion of this subsystem and a remaining neutron<sup>3</sup>.  $N_{\nu,k}^l$  is the normalization coefficient.

For  $\nu \gg k$  variables in (8) are separated and the latter set of equations is representable in the form of a sum of two sets of equations. One of them describes the relative motion of  ${}^{A-1}X$  subsystem and a neutron:

$$\begin{aligned} & -\frac{1}{4} \sqrt{(2\nu - k - l)(2\nu - k + l + 1)} C_{\nu-1,k}^l + \frac{1}{2} \left( 2\nu - k + \frac{3}{2} - 2\tilde{E} \right) C_{\nu,k}^l - \\ & -\frac{1}{4} \sqrt{(2\nu - k - l + 2)(2\nu - k + l + 3)} C_{\nu+1,k}^l = 0, \end{aligned} \tag{9}$$

while the other one characterizes the  ${}^{A-1}X = {}^{A-2}X + n$  subsystem itself:

<sup>3</sup> Recall that only the states with total angular momentum  $L = 0$  are considered in this paper.

$$\begin{aligned}
& - \sqrt{\frac{\lambda_{k-2}}{\lambda_k}} \cdot \frac{1}{4} \sqrt{(k-l)(k+l+1)} C_{\nu-1,k-2}^l + \frac{1}{2} \left( k + \frac{3}{2} - 2\varepsilon \right) C_{\nu,k}^l - \\
& - \sqrt{\frac{\lambda_k}{\lambda_{k+2}}} \cdot \frac{1}{4} \sqrt{(k-l+2)(k+l+3)} C_{\nu+1,k+2}^l = 0. \tag{10}
\end{aligned}$$

We here put the nucleon mass, the Planck's constant and oscillator length equal to 1 for the sake of brevity.

Thus asymptotically the three-cluster Schrödinger equation (8) can be reduced to a two-body-like multichannel problem. Total energy  $E$  is distributed between energy  $\varepsilon$  of the two-cluster subsystem  ${}^{A-1}\text{X}$  and energy  $\tilde{E}$  of the relative motion of the above-mentioned subsystem and a remaining neutron so that  $\tilde{E} + \varepsilon = E$ . Note that Eq. (9) corresponds to the free motion of a neutron and  ${}^{A-1}\text{X}$  subsystem, while the matrix of Eq. (9) contains the limit eigenvalues  $\lambda_k$  of the allowed states and, for this reason, is not identical to the matrix corresponding to the kinetic energy operator of free motion of the  ${}^{A-2}\text{X}$  cluster and a neutron. In other words, it contains some effective cluster-cluster interaction derived from the kinetic-energy operator modified by the Pauli principle. It is precisely this interaction that causes the  ${}^A\text{X}$  system to decay via an intermediate stage, i.e.,  ${}^A\text{X} \rightarrow {}^{A-1}\text{X} + n \rightarrow {}^{A-2}\text{X} + n + n$ . Alternative decay channel  ${}^A\text{X} \rightarrow {}^{A-2}\text{X} + {}^2n \rightarrow {}^{A-2}\text{X} + n + n$  is much less favorable, because in this case Eq. (10) would describe just a free motion of two neutrons and, thus, a sufficiently high energy is needed to make this decay channel open. The possibility for such decay channel to exist in  ${}^5\text{H}$  is discussed in Section 5.

From the above discussion it appears that the greater is the number of the families of the Pauli-allowed states invoked, the better is the description of the  ${}^{A-1}\text{X}$  subsystem and the less is its energy  $\varepsilon$ . Failure to take into account a sufficient number of families results in too localized  ${}^{A-1}\text{X}$  subsystem and, hence, in too high energy  $\varepsilon$ . Noteworthy also is the fact that in the limit of the large number of oscillator quanta  $\nu$  the kinetic energy operator does not couple the states belonging to the families corresponding to even values of quantum number  $k$  with those characterized by odd-numbered  $k$ . Such phenomenon takes place because even and odd values of  $k$  correspond to the states of the  ${}^{A-1}\text{X}$  subsystem with different parity, which becomes an integral of motion at large distance between the  ${}^{A-1}\text{X}$  subsystem and a neutron.

Of course, the cluster-cluster interaction generated by the nucleon-nucleon potential should also participate in Eq. (9) providing for realistic description of the  ${}^{A-1}\text{X}$  subsystem. However, inclusion of the nucleon-nucleon interaction between nucleons belonging to different clusters could not change the structure of the Pauli-allowed states and in a qualitative sense affect asymptotic behavior of the expansion coefficients of the wave function over these states. Study of the effects related to the potential energy (with its exchange part) in the microscopic analytical approach described here is under its way, and the results will be published in a separate paper.

Difference equations (9) and (10) become Bessel equations in the limit of large  $\nu$  and  $k$ :

$$\left( \frac{d^2}{dy_\nu^2} + \frac{1}{y_\nu} \frac{d}{dy_\nu} + 2\tilde{E} - \frac{(2l+1)^2}{4y_\nu^2} \right) C_k^l(y_\nu) = 0; \quad y_\nu = \sqrt{4\nu - 2k + 3}. \tag{11}$$

$$\left( \frac{d^2}{dy_k^2} + \frac{1}{y_k} \frac{d}{dy_k} + 2\varepsilon - \frac{(2l+1)^2}{4y_k^2} \right) C_\nu^l(y_k) = 0; \quad y_k = \sqrt{2k + 3}. \tag{12}$$

Therefore, the asymptotic form of the expansion coefficients in the  $l$ -basis can be conveniently written in terms of the Hankel functions  $H_{l+1/2}^\pm$  and the scattering  $S$ -matrix elements. If the incoming wave is in the channel characterized by angular momentum  $l$  and energy  $E - \varepsilon_i$  of the relative motion of  ${}^{A-1}\text{X}$  subsystem and a neutron, the expansion coefficients in this channel satisfy the asymptotic relation

$$\begin{aligned}
C_{\nu,k}^l(E - \varepsilon_i) &= H_{l+1/2}^{(-)} \left( \sqrt{2(E - \varepsilon_i)} \sqrt{4\nu - 2k + 3} \right) H_{l+1/2}^{(-)} \left( \sqrt{2\varepsilon_i} \sqrt{2k + 3} \right) - \\
&- \sum_{j=1}^{k_{\max}} \mathbf{S}_{ij} H_{l+1/2}^{(+)} \left( \sqrt{2(E - \varepsilon_j)} \sqrt{4\nu - 2k + 3} \right) H_{l+1/2}^{(+)} \left( \sqrt{2\varepsilon_j} \sqrt{2k + 3} \right).
\end{aligned}$$

Here index  $i$  enumerates different decay channels and its maximum possible value is equal to the number of different families considered. We emphasize that energy levels  $\varepsilon_i$  of the  ${}^{A-1}X$  subsystem do not necessarily belong to the discrete states, but can be continuum states as well. Even in the latter case it is possible to indicate the most favorable decay channels. For example, the  ${}^{A-1}X$  subsystem may not have any bound states, but could have a low-energy resonance. In such a case the  ${}^{A-2}X + n + n$  three-cluster system would most likely decay via resonance state of the  ${}^{A-1}X$  subsystem.

## 5. The Pauli-allowed states of the ${}^3H+n + n$ system

Let us use a three-cluster configuration  ${}^3H+n + n$  of the  ${}^5H$  nucleus with  $J^\pi = 1/2^+$  to illustrate the validity of our conclusions. We shall focuss our attention on the states with positive parity, zero total orbital momentum and zero spin of the two-neutron subsystem. For the sake of brevity the above-mentioned quantum numbers will be dropped in what follows.

In this case the norm kernel (2) takes the form:

$$I = \cosh(\mathbf{a}\tilde{\mathbf{a}}) \cosh(\mathbf{b}\tilde{\mathbf{b}}) - \cosh(\mathbf{a}_+\tilde{\mathbf{a}}) \cosh(\mathbf{b}_+\tilde{\mathbf{b}}) - \cosh(\mathbf{a}_-\tilde{\mathbf{a}}) \cosh(\mathbf{b}_-\tilde{\mathbf{b}}) + \cosh\{(\mathbf{a}_0\tilde{\mathbf{a}})\}, \quad (13)$$

where  $\mathbf{a}_\pm, \mathbf{b}_\pm$  and  $\mathbf{a}_0$  are defined by Eq. (4) with  $t = 5/3$ .

The norm kernel (13) have to be projected to the states with definite indices of the SU(3) symmetry  $(2\nu-4\mu, 2\mu)$  and expanded over a complete set of the eigenstates  $\Psi_{(2\nu-4\mu, 2\mu)_k}(\mathbf{a}, \mathbf{b})$  of the antisymmetrization operator:

$$I_{(2\nu-4\mu, 2\mu)} = \sum_{k=2\mu}^{\nu} \Lambda_{(2\nu-4\mu, 2\mu)_k} \Psi_{(2\nu-4\mu, 2\mu)_k}.$$

Eigenvalues of the norm kernel can be arranged into the branches and families quite as discussed in Section 3. Eigenvalues belonging to the first five families are given in Table 1.

Table 1  
Eigenvalues  $\Lambda_{(2\nu-4\mu, 2\mu)_k}$  of the norm kernel for the system  ${}^3H+n + n$

$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$\nu$	$\mu = 0$	$\mu = 0$	$\mu = 1$	$\mu = 1$
1	1.8889			
2	1.4780	0.9541	0.7778	
3	1.3487	0.8814	0.8733	1.0895
4	1.3344	0.8862	0.8879	1.0636
5	1.3334	0.8885	0.8888	1.0431
6	1.3333	0.8888	0.8888	1.0378
7	1.3333	0.8889	0.8889	1.0371
8	1.3333	0.8889	0.8889	1.0370
9	1.3333	0.8889	0.8889	1.0370
10	1.3333	0.8889	0.8889	1.0370

$\nu$	$\mu = 0$	$\mu = 1$	$\mu = 0$	$\mu = 1$	$\mu = 0$	$\mu = 1$	$\mu = 2$
1	1.8889						
2	1.4780	0.9541	0.7778				
3	1.3487	0.8814	0.8733	1.0895	1.0636	1.0431	0.9837
4	1.3344	0.8862	0.8879	1.0636	1.0434	1.0161	0.9837
5	1.3334	0.8885	0.8888	1.0431	1.0377	0.9841	0.9847
6	1.3333	0.8888	0.8888	1.0378	1.0371	0.9861	0.9870
7	1.3333	0.8889	0.8889	1.0371	1.0370	0.9873	0.9876
8	1.3333	0.8889	0.8889	1.0370	1.0370	0.9876	0.9876
9	1.3333	0.8889	0.8889	1.0370	1.0370	0.9876	0.9876
10	1.3333	0.8889	0.8889	1.0370	1.0370	0.9876	0.9876

One can readily see from Table 1 that the  ${}^5H$  norm kernel eigenvalues belonging to the  $k$ th family tend to the eigenvalues of a two-cluster subsystem  ${}^4H$  with increasing the number of oscillator quanta  $\nu$ :

$$\lim_{\nu-2\mu \rightarrow \infty} \Lambda_{(2\nu-4\mu, 2\mu)_k}^{{}^5H={}^3H+n+n} \rightarrow \lambda_k^{{}^4H={}^3H+n} = 1 - \left(-\frac{1}{3}\right)^k.$$

Obviously, index  $k$  makes sense of the number of oscillator quanta accounted for by the  ${}^4\text{H}$  subsystem. Of special note is the fact that  $\lambda_{k=2\tilde{k}+1} > 1$ , while  $\lambda_{k=2\tilde{k}} < 1$ . This in itself is indicative of attraction between  ${}^3\text{H}$  and a neutron in the states with odd number of quanta  $k = 2\tilde{k} + 1$  and repulsion in the states with even number of quanta  $k = 2\tilde{k}$ . Such a conclusion can be easily verified by analyzing the asymptotic Eq. (10). Moreover, the eigenvalues  $\Lambda_{(2\nu-4\mu,2\mu)_{2\tilde{k}+1}}$  approach limit eigenvalues  $\lambda_{2\tilde{k}+1}$  from above, contrastingly to  $\Lambda_{(2\nu-4\mu,2\mu)_{2\tilde{k}}}$  approaching  $\lambda_{2\tilde{k}}$  from below. As was concluded in Section 3, there are strong grounds for believing that families of the states characterized by odd values of quantum number  $k$  dominates in the wave function of the  ${}^5\text{H}$  system. Table 2 illustrates the fact that the Pauli-allowed states  $\Psi_{(2\nu-4\mu,2\mu)_k}$  take

Table 2

Overlap integrals  $\int \Psi_{(2\nu-4\mu,2\mu)_k} \Psi_{(2\nu-4\mu,2\mu)_k}^{\text{as}} d\mu_B$  versus the number of quanta  $\nu$  for the first five families of the Pauli-allowed states of the  ${}^5\text{H}$ .

	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$						
$\nu$	$\mu = 0$	$\mu = 0$	$\mu = 1$	$\mu = 0$	$\mu = 1$	$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 0$	$\mu = 1$	$\mu = 2$
1	0.9701										
2	0.9460	0.9924	1								
3	0.9893	0.9979	0.9934	0.8076	0.9915						
4	0.9996	0.9998	0.9992	0.9007	0.9886	0.9214	0.9629	1			
5	1.0000	1.0000	1.0000	0.9736	0.9983	0.9706	0.9900	0.9934	0.5887	0.9052	0.9915
6	1.0000	1.0000	1.0000	0.9981	0.9999	0.9962	0.9979	0.9992	0.7299	0.9907	0.9886
7	1.0000	1.0000	1.0000	0.9999	1.0000	0.9995	0.9998	1.0000	0.9294	0.9974	0.9983
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9932	0.9996	0.9999
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	1.0000	1.0000
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

asymptotic form

$$\Psi_{(2\nu-4\mu,2\mu)_k}^{\text{as}} = \frac{1}{\sqrt{2}} \left( \psi_{(2\nu-4\mu,2\mu)}^{k-2\mu}(\mathbf{a}_1, \mathbf{b}_1) + \psi_{(2\nu-4\mu,2\mu)}^{k-2\mu}(\mathbf{a}_2, \mathbf{b}_2) \right)$$

as  $\Lambda_{(2\nu-4\mu,2\mu)_k}$  approach their limit eigenvalues. In Table 2 overlap integrals of the exact Pauli-allowed states with their asymptotic expressions  $\Psi_{(2\nu-4\mu,2\mu)_k}^{\text{as}}$  depending on the number of quanta  $2\nu$  are given for the first five families of the  ${}^5\text{H}$  norm kernel eigenfunctions. As long as  $\nu \leq 5$ , the states listed in Table 2 exhaust all possible basis functions allowed by the Pauli principle. As  $\nu$  increases, new families of states characterized by  $k > 5$  also appear. However, eigenstates and eigenvalues belonging to such families are governed by the same laws as those tabulated in Table 2. Furthermore, states of "odd" families, i.e.,  $\Psi_{(2\nu-4\mu,2\mu)_{2\tilde{k}+1}}$  appear to dominate over the states  $\Psi_{(2\nu-4\mu,2\mu)_{2\tilde{k}}}$ , with small  $\tilde{k}$  prevailing. The first family  $\Psi_{(2\nu,0)_{k=1}}$  plays a leading part in the wave function of the  ${}^5\text{H}$  system, followed by the families  $\Psi_{(2\nu-4\mu,2\mu)_{k=3}}$  and  $\Psi_{(2\nu-4\mu,2\mu)_{k=5}}$ . Such hierarchy among different families of the Pauli-allowed states of the  ${}^5\text{H}$  is directly related to the behavior of the corresponding eigenvalues  $\Lambda_{(2\nu-4\mu,2\mu)_{2\tilde{k}+1}}$  discussed above. An exact treatment of the antisymmetrization effects related to the kinetic energy exclusively was shown to result in an effective attraction of the clusters in those branches whose eigenvalues exceed their limiting values.

From Table 1 and Table 2 we can observe that a complete space of the Pauli-allowed basis functions can be divided into "internal" region, where  $\nu - k < 5$ , and asymptotic region ( $\nu - k \geq 5$ ). In the latter region functions  $\Psi_{(2\nu-4\mu,2\mu)_k}$  take simple analytical form  $\Psi_{(2\nu-4\mu,2\mu)_k}^{\text{as}}$  and correspond to the decay of the  ${}^5\text{H}$  nucleus into the  ${}^4\text{H}$  subsystem and a neutron, with the  ${}^4\text{H}$  cluster occurring in the state with  $k$  oscillator quanta. In the internal region the Pauli-allowed states have more complicated form, which is indicative of exchange effects involving all three clusters.

The states of the  $k$ th family have one more important feature, which makes small  $k$  predominating. The less is  $k$ , the more smooth basis functions  $\Psi_{(2\nu-4\mu,2\mu)_k}$  are. Let us demonstrate this with the functions

belonging to the first family consisting of a single branch  $\Psi_{(2\nu-4\mu,2\mu)_{k=1}}$ . The latter branch has a remarkably simple asymptotic form:

$$\Psi_{(2\nu,0)_1}^{\text{as}} = \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{2\nu}{(2\nu+1)!}} \left\{ \mathbf{a}_1^{2\nu-2}(\mathbf{a}_1 \mathbf{b}_1) + \mathbf{a}_2^{2\nu-2}(\mathbf{a}_2 \mathbf{b}_2) \right\},$$

and the exact function  $\Psi_{(2\nu-4\mu,2\mu)_{k=1}}$  becomes indistinguishable from the asymptotic one at  $\nu = 5$ . It is easily comprehended that function  $\Psi_{(2\nu,0)_1}^{\text{as}}$  corresponds to the decay of the  ${}^5\text{H}$  nucleus into the  ${}^4\text{H}$  subsystem being in the lowest oscillator shell model state and a neutron. Indeed,  $\Psi_{(2\nu,0)_1}^{\text{as}}$  contains only the first power of vector  $\mathbf{b}_1$  or  $\mathbf{b}_2$  evidencing that  ${}^3\text{H}$  cluster and a neutron remain at the minimal distance from each other, which is consistent with the requirements of the Pauli exclusion principle. In turn, eigenvalues of the first family  $\Lambda_{(2\nu,0)_1}$  tend to the lowest eigenvalue of the  ${}^4\text{H}$  norm kernel  $\lambda_{k=1} = 4/3$ . It is noteworthy also that the orbital angular momentum  $l$  of the relative motion of the neutron cluster and  ${}^4\text{H}$  subsystem, as well as the orbital momentum of the  ${}^4\text{H}$  itself is  $l = 1$  for the states of the first branch.

As noted in Section 3, at  $\nu \geq k+5$  the expansion coefficients  $D_{(2\nu-4\mu,2\mu)_k}^{2m-2\mu}$  of the Pauli-allowed basis functions  $\Psi_{(2\nu-4\mu,2\mu)_k}$  can be identified with the Kravchuk polynomials of a discrete variable  $2m-2\mu$ :

$$D_{(2\nu-4\mu,2\mu)_k}^{2m-2\mu} \rightarrow \sqrt{2} \mathcal{K}_{k-2\mu}^{(5/8)}(2m-2\mu) \frac{\sqrt{\rho_{2m-2\mu}}}{d_{k-2\mu}}.$$

Dependence of the expansion coefficients of the first branch  $\Psi_{(2\nu-4\mu,2\mu)_{k=1}}$  on the number of quanta  $\nu$  is shown in Fig. 1. At a given  $\nu$  there are  $\nu+1$  coefficients  $D_{(2\nu,0)_{k=1}}^{2m}$ . As expected, the latter coefficients

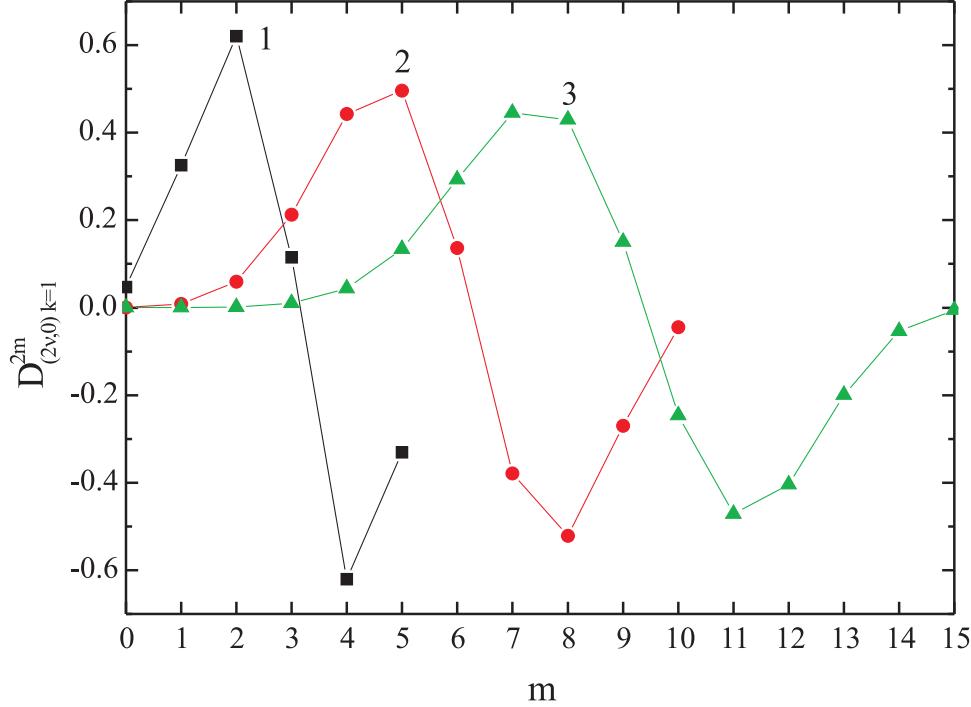


Fig. 1. Expansion coefficients  $D_{(2\nu,0)_{k=1}}^{2m}$  of the first branch versus  $m$  at different values of the number of quanta  $\nu$ . Curves: 1 –  $\nu = 5$ ; 2 –  $\nu = 10$ ; 3 –  $\nu = 15$ .

have one node at all  $\nu$ , because their asymptotic behavior is determined by the Kravchuk polynomials of the first order. Note that with increase in  $\nu$  coefficients  $D_{(2\nu,0)_{k=1}}^{2m}$  become Hermitian polynomials of the first order as follows from (A.1). Evidently, the less is the number of nodes, the lower is the kinetic energy and

potential energy corresponding to a given branch. Therefore, families of the allowed states associated with the low-order Kravchuk polynomials are more favorable.

It should be pointed out that amongst eigenfunctions of the antisymmetrization operator of the  ${}^5\text{H}$  system belonging to SU(3) representation  $(2\nu, 0)$  there is a nodeless branch of eigenfunctions  $\Psi_{(2\nu, 0)_{k=0}}$ . However, it falls into the category of the Pauli-forbidden states and corresponds to zero eigenvalue. Functions of this branch can be written in analytical form for a given  $\nu$ :

$$\Psi_{(2\nu, 0)_{k=0}} = N_{(2\nu, 0)_{k=0}} \cdot \frac{1}{\sqrt{(2\nu + 1)!}} \{ \mathbf{a}_1^{2\nu} + \mathbf{a}_2^{2\nu} \},$$

where  $N_{(2\nu, 0)_{k=0}}$  is the normalization coefficient, which tends to  $1/\sqrt{2}$  with  $\nu$  increasing. The expansion coefficients  $D_{(2\nu, 0)_{k=0}}^{2m}$  are given by the following expression:

$$D_{(2\nu, 0)_{k=0}}^{2m} = \left(\frac{3}{8}\right)^{\nu} \sqrt{\frac{(2\nu)!}{(2\nu - 2m)!(2m)!}} \left(\frac{5}{3}\right)^m.$$

The structure of  $\Psi_{(2\nu, 0)_{k=0}}$  indicates that one of the valence neutrons occupies an  $s$ -state in the  ${}^4\text{H}$  subsystem, which is forbidden by the Pauli principle.

Let us discuss now possibility for another decay channel of the  ${}^5\text{H}$  to exist. At first glance it would seem that  ${}^5\text{H} \rightarrow {}^2\text{n} + {}^3\text{H} \rightarrow n + n + {}^3\text{H}$  decay should occur along with the  ${}^5\text{H} \rightarrow n + {}^4\text{H} \rightarrow n + n + {}^3\text{H}$  decay. However, the structure of the Pauli-allowed functions is not compatible with such assumption. Basic functions corresponding to the "dineutron" decay of the  ${}^5\text{H}$ , if any, should look like

$$\Psi_{(2\nu - 4\mu, 2\mu)_{2m}}(\mathbf{a}, \mathbf{b}) \rightarrow \psi_{(2\nu - 4\mu, 2\mu)}^{2m-2\mu}(\mathbf{a}, \mathbf{b}), \quad \mu \leq m \leq \nu - \mu. \quad (14)$$

with eigenvalues

$$\lim_{\nu - 2\mu \rightarrow \infty} \Lambda_{(2\nu - 4\mu, 2\mu)_{2m}} \rightarrow \lambda_{2m}^{2n=n+n} = 1.$$

Asymptotic behavior of the simplest function relevant to the decay of the  ${}^5\text{H}$  into the dineutron being in the lowest shell-model state and triton is of the form

$$\Psi_{(2\nu, 0)_{m=0}}^{\text{as}} = \frac{1}{\sqrt{(2\nu + 1)!}} \mathbf{a}^{2\nu}. \quad (15)$$

From this it readily follows that

$$\lim_{\nu \rightarrow \infty} D_{(2\nu, 0)_{m=0}}^{2m} \rightarrow \delta_{m,0}.$$

In Table 3 eigenvalues of those basis states which most closely resemble functions (15) are listed along with the weight of the state  $\psi_{(2\nu, 0)}^{m=0}(\mathbf{a}, \mathbf{b})$ .

Table 3

Eigenvalues of the candidate for the "dineutron decay" function  $\Psi_{(2\nu, 0)_{m=0}}$  of the  ${}^5\text{H}$  and overlap  $D_{(2\nu, 0)}^{m=0} = \int \Psi_{(2\nu, 0)_{m=0}}^{\text{as}} \Psi_{(2\nu, 0)_{m=0}} d\mu_B$  versus number of quanta  $\nu$ .

$\nu$	1	2	3	4	5	6	7	8	9	10
$\Lambda_{(2\nu, 0)_{m=0}}$	1.8889	1.4781	1.0895	1.0636	1.0139	1.0087	1.0027	1.0015	1.0008	1.0003
$D_{(2\nu, 0)}^{m=0}$	0.8575	0.7304	0.8407	0.7327	0.8564	0.8510	0.8141	0.9518	0.8237	0.9406

As may be inferred from this table, the dependence of the coefficient  $D_{(2\nu, 0)}^{m=0}$  from the number of quanta  $\nu$  is not monotone and even at  $\nu = 10$  the weight of the "dineutron decay" function is only 88%. Note that the weight of the simplest  ${}^4\text{H} + n$  function in the states of the first family runs up to 100% at  $\nu = 5$ . As regards the branches  $\Psi_{(2\nu, 0)_{2m>0}}^{\text{as}}$ , which describe the decay of the  ${}^5\text{H}$  into the dineutron being in the state with

$2m$  quanta and a triton, none of the Pauli-allowed states of the  ${}^5\text{H}$  has such asymptotic behavior. Figure 2 depicts the expansion coefficients  $D_{(2\nu,0)m=0}^{2m}$  of the branch, which is a candidate for the "dineutron decay" mode, at different values of the number of quanta  $\nu$ . As evident from Fig. 2, coefficients  $D_{(2\nu,0)m=0}^{2m}$  have

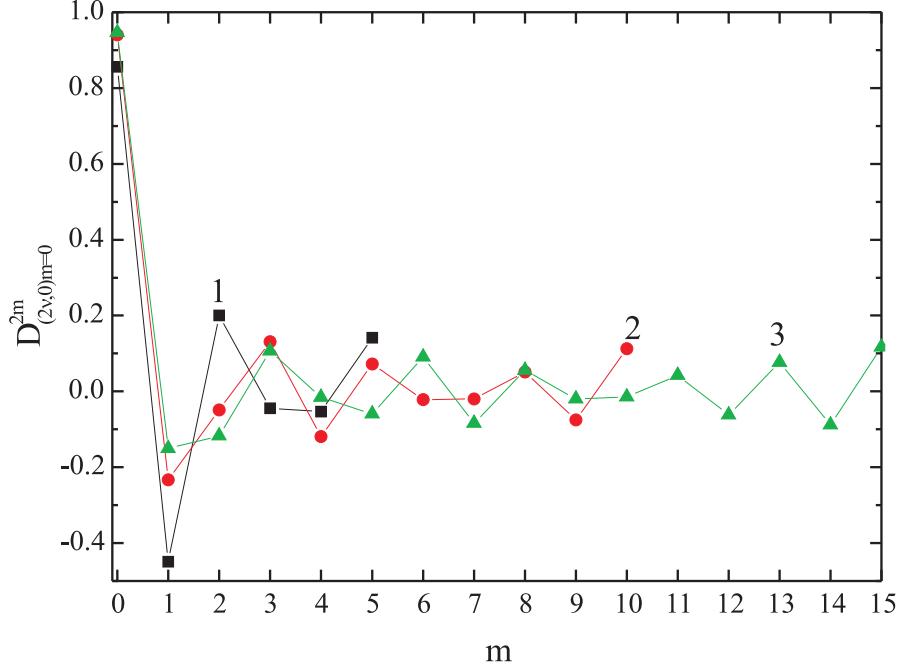


Fig. 2. Expansion coefficients  $D_{(2\nu,0)m=0}^{2m}$  of the "dineutron decay" branch versus  $m$  at different values of the number of quanta  $\nu$ . Curves: 1 –  $\nu = 5$ ; 2 –  $\nu = 10$ ; 3 –  $\nu = 15$ .

noticeable tail even at  $\nu = 15$ , indicating that function  $\Psi_{(2\nu,0)m=0}$  still does not coincide with its asymptotic expression  $\Psi_{(2\nu,0)m=0}^{\text{as}}$ . Furthermore, it has too many nodes and thus quite high energy is needed to excite such a mode. All the foregoing counts in favour of the conclusion that  ${}^5\text{H} \rightarrow {}^2\text{n} + {}^3\text{H} \rightarrow n + n + {}^3\text{H}$  decay is not realized. As for the states  $\Psi_{(2\nu-4\mu,2\mu)_k}$ , which are dominated by the component  $\psi_{(2\nu,0)}^{m=\mu}(\mathbf{a}, \mathbf{b})$ , they may be thought of as exhibiting some  $nn$  correlations in the internal region  $\nu - k < 5$ .

In regard to the "democratic decay" of the  ${}^5\text{H}$ , it is highly improbable due to the fact that all limit eigenvalues of the  ${}^5\text{H}$  nucleus, namely  $\lambda_k^{{}^4\text{H}}$ , are different. Distinction of limit eigenvalues implies that there is no region where the Pauli-allowed states of the three-cluster system  ${}^5\text{H}$  coincide with hyperspherical harmonics, which reproduce the "democratic decay". Finally, Figure 3 demonstrates the coefficients  $C_{(2\nu-4\mu,2\mu)_k}^E$  appearing in the expansion of the  ${}^5\text{H}$  wave function  $\Upsilon_{\kappa(E)}(\mathbf{a}, \mathbf{b})$  with energy  $E = 10.67$  MeV above the  ${}^5\text{H} \rightarrow {}^3\text{H} + n + n$  decay threshold in the basis of the Pauli-allowed states  $\Psi_{(2\nu-4\mu,2\mu)_k}$ . These coefficients have been obtained by diagonalization of the Hamiltonian, in which only the operator of the kinetic energy of the relative motion (with its exchange part) is retained. First five families were taken into complete account, families with  $k > 5$  were disregarded. Oscillator length  $r_0$  was chosen so to reproduce root-mean-square radius of the  ${}^3\text{H}$  cluster. In fact, the eigenfunctions of the kinetic energy operator modified by the Pauli principle demonstrate the manner in which the antisymmetrization influences the expansion coefficients  $C_{(2\nu-4\mu,2\mu)_k}^E$ . Figure 3 sustains our conclusion that families characterized by odd values of  $k$  contribute much more significantly to the wave function of the  ${}^5\text{H}$  than families with even  $k$  do. An hierarchy among the coefficients of different branches of a given family is established by the behavior of their eigenvalues, namely, by the magnitude of attraction or repulsion in the corresponding SU(3) branches. By this reason, the coefficients of the branches  $(2\nu, 0)_{k=1}$  take the lead, followed by the  $(2\nu, 0)_{k=3}$  and  $(2\nu - 4, 2)_{k=3}$ . Neglecting families of the states with  $k = 2$  and  $k = 4$  does not change considerably the wave function of the

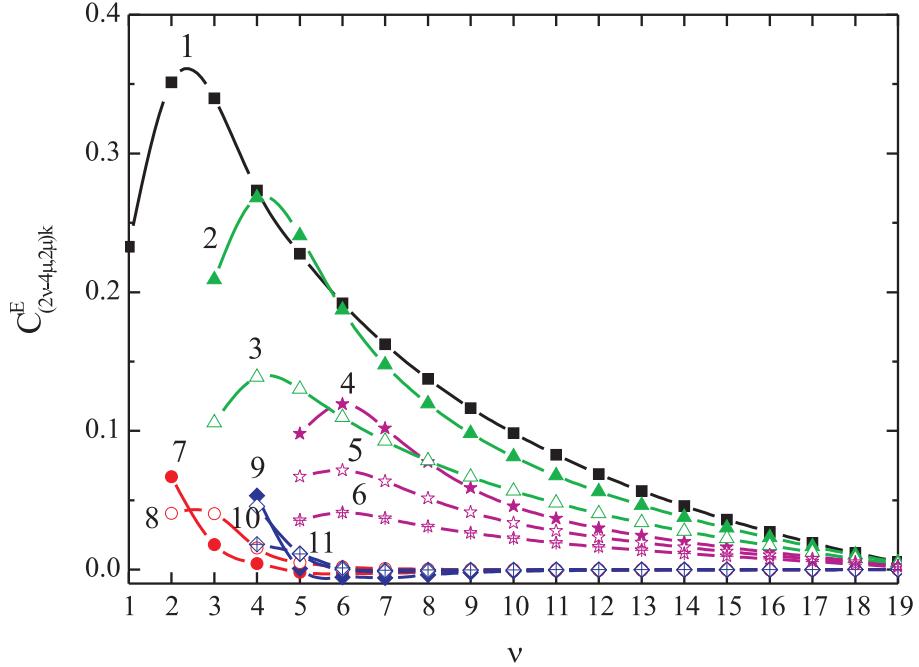


Fig. 3. Coefficients  $C_{(2\nu-4\mu,2\mu)k}^E$  of the expansion of the  ${}^5\text{H}$  wave function in the SU(3)-basis at  $E = 10.67$  MeV. Curves: 1 –  $(2\nu, 0)_{k=1}$ ; 2 –  $(2\nu, 0)_{k=3}$ ; 3 –  $(2\nu - 4, 2)_{k=3}$ ; 4 –  $(2\nu, 0)_{k=5}$ ; 5 –  $(2\nu - 4, 2)_{k=5}$ ; 6 –  $(2\nu - 8, 4)_{k=5}$ ; 7 –  $(2\nu, 0)_{k=2}$ ; 8 –  $(2\nu - 4, 2)_{k=2}$ ; 9 –  $(2\nu, 0)_{k=4}$ ; 10 –  $(2\nu - 4, 2)_{k=4}$ ; 11 –  $(2\nu - 8, 4)_{k=4}$ .

${}^5\text{H}$  and the energy  $E$ . The latter takes such a large value mainly due to the fact that energy  $\varepsilon$  of the  ${}^4\text{H}$  subsystem is high enough at given localization of the  ${}^4\text{H}$ . Of course, taking into account nucleon-nucleon interaction between  ${}^3\text{H}$  cluster and a neutron would reduce this energy. Involvement of the "odd" families with higher  $k$  also leads to better description of the  ${}^4\text{H}$  subsystem and, hence, to lowering the energy  $E$ . It should be also pointed out that the behavior of the expansion coefficients  $C_{(2\nu-4\mu,2\mu)k}^E$  provides support for the predominance of the decay  ${}^5\text{H} \rightarrow n + {}^4\text{H} \rightarrow n + n + {}^3\text{H}$ .

## 6. Conclusion

New approach to the problem of multichannel continuum spectrum of  ${}^A\text{X} = {}^{A-2}\text{X} + n + n$  systems ( $A \leq 6$ ) is suggested based on the discrete representation of a complete basis of allowed states of the multiparticle harmonic oscillator, which was systematized with the help of the indices of SU(3) symmetry and defined in the Fock-Bargmann space. Proposed approach allows correct description of three-cluster systems both in the region of small intercluster distances, where the Pauli exclusion principle is of first importance, and in the asymptotic region where the scattering matrix elements are produced.

Careful analysis of the structure of the eigenfunctions and behavior of the eigenvalues of the three-cluster norm kernel has been performed for the first time. A set of linear algebraic equations which generates the Pauli-allowed states for different three-cluster systems composed of an  $s$ -cluster and two neutrons is written in a general form. The latter depends on the single parameter which can be easily found for any three-cluster system.

In the Fock-Bargmann space the Pauli-allowed states of a three-cluster system are shown to be superpositions of the hypergeometric functions, with the expansion coefficients being orthogonal polynomials of a discrete variable. Eigenvalues of the three-cluster system are shown to tend to eigenvalues of a two-cluster subsystem with increasing the number of oscillator quanta  $\nu$ . At the same time, corresponding eigenvectors take simple analytical form, while the expansion coefficients become the Kravchuk polynomials as the num-

ber of oscillator quanta increases. We suggest a way of resolving the problem of the SU(3) degeneracy of the Pauli-allowed states. A degree of the Kravchuk polynomial serves as an additional quantum number to label the states belonging to the same SU(3) irreducible representations.

The Pauli-allowed states of a three-cluster system  ${}^{A-2}X + n + n$  can be arranged into branches and families, with all the states of a particular branch having common SU(3)-symmetry index  $\mu$ , but differing in value of the first SU(3) index  $\lambda$ . The eigenvalues belonging to a given branch tend to the same eigenvalue of the two-body subsystem  $\lambda_k^{{}^{A-1}X}$  with the number of quanta increasing. The branches which share limit eigenvalues are combined in the family of the eigenstates, which thus is completely determined by the degree  $k$  of the corresponding Kravchuk polynomial. Each family of the Pauli-allowed states asymptotically corresponds to a certain binary decay channel of a three-cluster system into a two-cluster subsystem  ${}^{A-1}X$  occurring in a ground or an excited harmonic-oscillator state and a remaining neutron. Such asymptotic behavior gives an indication of possible decay channels of a three-cluster nucleus and allows us to specify the most important decay channels of the nucleus under consideration.

The families which characterized by eigenvalues approaching limit values  $\lambda_k$  from above are shown to be more favorable at small  $\nu$ , which can be considered as effective attraction of clusters due to the exchange effects. It is precisely this interaction that causes the  ${}^A X$  system to decay via an intermediate stage, i.e.,  ${}^A X \rightarrow {}^{A-1} X + n \rightarrow {}^{A-2} X + n + n$ . Alternative decay channel  ${}^A X \rightarrow {}^{A-2} X + {}^2 n \rightarrow {}^{A-2} X + n + n$  is not realized.

It is observed that distinction of limit eigenvalues of the three-cluster norm kernel results in the absence of such region where the Pauli-allowed states of the  ${}^{A-2}X + n + n$  system coincide with hyperspherical harmonics, which reproduce the "democratic decay". Hence, instead of hyperspherical functions the angular-momentum basis functions, which are labeled by the number of quanta, the angular momentum of the  ${}^{A-1}X$  subsystem and momentum of the relative motion of this subsystem and a remaining neutron, should be used in the limit  $\nu \gg k$ . As the result, asymptotically the three-cluster Schrödinger equation can be reduced to a two-body-like multichannel problem, while the asymptotic form of the expansion coefficients in the  $l$ -basis is expressed in terms of the Hankel functions and the scattering  $S$ -matrix elements.

The validity of these conclusions was illustrated with the  ${}^3H + n + n$  configuration of the  ${}^5H$  nucleus. In particular, the structure of the Pauli-allowed states of the  ${}^5H$  nucleus was revealed to correspond to the subsequent decay  ${}^5H \rightarrow n + {}^4H \rightarrow n + n + {}^3H$ .

## Appendix A. The Kravchuk polynomials

The Kravchuk polynomial  $\mathcal{K}_k^{(p)}(m)$  of a discrete variable  $m$ , specified on the interval  $0 \leq m \leq \nu$  and orthogonal with weighting function  $\rho_m$  and norm  $d_k$

$$\rho_m = \frac{\nu!}{m!(\nu - m)!}, \quad d_k = \sqrt{\frac{\nu!}{k!(\nu - k)!}} (pq)^{k/2}, \quad q = 1 - p,$$

can be defined via hypergeometric function:

$$\begin{aligned} \bar{\mathcal{K}}_k^{(p)}(m) &\equiv \mathcal{K}_k^{(p)}(m) \cdot \frac{\sqrt{\rho_m}}{d_k} = \\ &= \begin{cases} \frac{(-1)^k}{(\nu - m - k)!} \sqrt{\frac{(\nu - k)!(\nu - m)!}{m!k!}} \cdot p^{\frac{m+k}{2}} q^{\frac{\nu - m - k}{2}} {}_2F_1 \left( -k, -m; \nu - m - k + 1; -\frac{q}{p} \right), & \text{if } m \leq \nu - k \\ \frac{(-1)^{m-\nu}}{(-\nu + m + k)!} \sqrt{\frac{m!k!}{(\nu - k)!(\nu - m)!}} \cdot q^{\frac{m+k-\nu}{2}} p^{\nu - \frac{m+k}{2}} {}_2F_1 \left( -\nu + k, -\nu + m; -\nu + m + k + 1; -\frac{q}{p} \right), & \text{if } m > \nu - k. \end{cases} \end{aligned}$$

Note that  $\nu$ ,  $m$  and  $k$  are the natural integers,  $0 \leq k \leq \nu$ .

Relation of orthogonality for the Kravchuk polynomials looks like:

$$\sum_{m=0}^{\nu} \bar{\mathcal{K}}_k^{(p)}(m) \bar{\mathcal{K}}_{\tilde{k}}^{(p)}(\tilde{m}) = \delta_{k,\tilde{k}}, \quad \sum_{k=0}^{\nu} \bar{\mathcal{K}}_k^{(p)}(m) \bar{\mathcal{K}}_{\tilde{k}}^{(p)}(\tilde{m}) = \delta_{m,\tilde{m}}.$$

Symmetry relation can be written in the following form:

$$(-1)^{m+k} \mathcal{K}_k^{(p)}(m) \cdot \frac{\sqrt{\rho_m}}{d_k} = \mathcal{K}_{\nu-k}^{(p)}(\nu-m) \cdot \frac{\sqrt{\rho_{\nu-m}}}{d_{\nu-k}}.$$

The Kravchuk polynomial has the limit

$$\lim_{\nu \rightarrow \infty} \mathcal{K}_k^{(p)}(m) \frac{\sqrt{\rho_m}}{d_k} = \frac{1}{\sqrt{2^k k!} \sqrt{\pi 2 p q \nu}} H_k \left( \frac{m - p \nu}{\sqrt{2 p q \nu}} \right) \exp \left\{ - \frac{(m - p \nu)^2}{4 p q \nu} \right\}, \quad (\text{A.1})$$

where  $H_k(x)$  is the Hermitian polynomial.

## References

- [1] A. A. Korsheninnikov, et al., Phys. Rev. Lett. 87 (2001) 092501.
- [2] M. S. Golovkov, et al., Phys. Lett. B 566 (2003) 70.
- [3] S. V. Stepanstsov, et al., Nucl. Phys. A 738 (2004) 436.
- [4] L. V. Chulkov, Nucl. Phys. A 734 (2004) 357.
- [5] Yu. B. Gurov, et al., Eur. Phys. J. A 24 (2005) 231.
- [6] N. B. Shulgina, B. V. Danilin, L. V. Grigorenko, M. V. Zhukov, J. M. Bang, Phys. Rev. C 62 (2000) 014312.
- [7] P. Descouvemont, A. Kharbach, Phys. Rev. C 63 (2001) 027001.
- [8] K. Arai, Phys. Rev. C 68 (2003) 034303.
- [9] R. de Diego, E. Garrido, D. V. Fedorov, A. S. Jensen, Nucl. Phys. A 786 (2007) 71.
- [10] J. Broeckhove, F. Arickx, P. Hellinckx, V. S. Vasilevsky and A. V. Nesterov, J. Phys. G: Nucl. Part. Phys. 34 (2007) 1955.
- [11] L. V. Grigorenko, Eur. Phys. J. A 20 (2004) 419.
- [12] D. R. Tilley, H. R. Weller, G. M. Hale, Nucl. Phys. A 541 (1992) 1.
- [13] V. I. Kukulin, V. G. Neudatchin, Yu. F. Smirnov, Fiz. Elem. Chastits At. Yadra 10 (1979) 1236.
- [14] H. Matsumura, M. Orabi, Y. Suzuki, Y. Fujiwara, Nucl. Phys. A 776 (2006) 1.
- [15] H. Horiuchi, Prog. Theor. Phys. 51 (1974) 1266.
- [16] G. Filippov, Yu. Lashko, Phys. Rev. C 70 (2004) 064001.
- [17] G. F. Filippov and Yu. A. Lashko, Phys. Part. Nucl., 36 (2005) 714.
- [18] Yu. A. Lashko, G. F. Filippov, Phys. Atom. Nucl., 70 (2007) 1017.
- [19] G. Filippov, Sov. J. Nucl. Phys., 33 (1981) 488.
- [20] P. Descouvemont, C. Daniel, D. Baye, Phys. Rev. C 67 (2003) 044309.
- [21] J. A. Wheeler, Phys. Rev. 32 (1937) 1083.
- [22] V. Bargmann, Ann. Math. 48 (1947) 568.
- [23] H. Horiuchi, Prog. Theor. Phys. 55 (1976) 1448.
- [24] Y. Fujiwara, H. Horiuchi, Prog. Theor. Phys. 63 (1980) 895.
- [25] G. F. Filippov, I. Yu. Rybkin, S. V. Korennov, J. Math. Phys. 36 (1995) 4571.
- [26] K. Katō, H. Bandō, Prog. Theor. Phys. 53 (1975) 692.
- [27] A. F. Nikiforov, S. K. Suslov, V. B. Uvarov, Classical Orthogonal Polynomials of a Discrete Variable (Springer-Verlag, Berlin-Heidelberg-New York, 1991).
- [28] G. F. Filippov, Yu. A. Lashko, S. V. Korennov, and K. Katō, Few-Body Syst. 34 (2004) 209.